Taylor and Maclaurin series (11.10)

5 Theorem If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!} \qquad c_0 = \frac{f^{(0)}(a)}{0!} = \frac{f(a)}{1}$$

$$f(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

This is the Taylor series/expansion of f at a.

Maclaurin series: this with $a = 0$. $\left\{ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \right.$

$$\underbrace{T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

$f(x) - T_n(x)$

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and if for all x in $(a-R, a+R)$

$$\lim_{n\to\infty} R_n(x) = 0,$$

~~for $|x-a| < R$~~, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

$(a-R, a+R)$

So: radius of convergence $\geq R$

0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

$$(1+x)^5 = \sum_{n=0}^{5} \binom{5}{n} x^n \qquad \binom{a}{b} = \frac{a!}{b!(a-b)!}$$

$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

$x \in (a-d, a+d)$

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d$$

This lets us compute approximations of $f(x)$ on an interval. Can help in applying Theorem 8

(Useful fact: $\lim_{n\to\infty} \frac{x^n}{n!} = 0$)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$R = \infty$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \text{for all } x$$

17 The Binomial Series If k is any real number and $|x| < 1$, then

Newton:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$\binom{a}{b} = \frac{a}{b}\binom{a-1}{b-1}$$

$$\binom{5}{2} = \frac{5}{2}\cdot\frac{4}{1}\cdot\binom{3}{0} = \frac{5}{2}\cdot\frac{4}{1} = 10$$

$$\binom{x}{2} = \frac{x}{2}\cdot\frac{(x-1)}{1}\cdot\binom{x}{0} = \frac{x(x-1)}{2}$$

$$\binom{\pi}{2} = \frac{\pi(\pi-1)}{2}$$

$$\sqrt{1+x} = (1+x)^{1/2} = \sum_{n=0}^{\infty}\binom{1/2}{n}x^n = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2 + \cdots$$

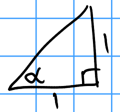
$$\begin{array}{r|l} & 1 + x + x^2 + x^3 + \cdots \\ 1 - x & 1 \\ & -(1 - x) \\ & \quad x \\ & \quad x - x^2 \\ & \qquad x^2 \\ & \qquad x^2 - x^3 \\ & \qquad\quad x^3 \end{array}$$

$$\tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt$$

$$= \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty}\int_0^x (-1)^n t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\alpha = \tan^{-1}(1) = \frac{\pi}{4}$$



$$\pi = 4\tan^{-1}(1) = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

$$= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

$f(x) = \sin(x)$ centered at 0

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f'''(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$

$f(0) = 0$
$f'(0) = 1$
$f''(0) = 0$
$f'''(0) = -1$
0
1
0
-1
$\vdots$

$$\text{series} = \overset{0}{1 \cdot \frac{x}{1!}} - \overset{1}{1 \cdot \frac{x^3}{3!}} + \overset{2}{1 \cdot \frac{x^5}{5!}} - \overset{3}{1 \cdot \frac{x^7}{7!}} + \overset{4}{1 \cdot \frac{x^9}{9!}} - \cdots$$

n	$2n$	$2n+1$	$2n-1$
0	0	1	(−1)
1	2	3	1
2	4	5	3
3	6	7	5

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (*)$$

$f^{(n)}(x) = \pm\sin(x)$ or $\pm\cos(x)$

$|f^{(n)}(x)| = |\sin(x)|$ or $|\cos x| \leq 1 = M$

(For all x, $|f^{(n)}(x)| \leq 1$.)

$$|R_n(x)| \leq \frac{M|x|^{n+1}}{(n+1)!} = \frac{|x|^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

then $\lim_{n\to\infty} |R_n(x)| = 0$ by squeeze thm.

By thm 8, $\sin(x) = (*)$ for <u>all</u> x.

ex $$\lim_{x\to 0} \frac{\sin(x) - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0} \frac{\left(\cancel{x} - \cancel{\frac{x^3}{6}} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) - \cancel{x} + \cancel{\frac{x^3}{6}}}{x^5} = \lim_{x\to 0}\left(\frac{1}{5!} - \frac{x^2}{7!} + \cdots\right)$$

$$= \frac{1}{5!}$$

ex $f(x) = \sqrt{x}$ $\quad a = 1$ $\quad$ @ $x=1$

	@ $x=1$	
$f(x) = \sqrt{x}$	1	
$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$	$\frac{1}{2}$	
$f''(x) = \frac{-1}{2\cdot 2}x^{-\frac{3}{2}}$	$\frac{-1}{2\cdot 2}$	$n=0$
$f'''(x) = \frac{3}{2\cdot 2\cdot 2}x^{-5/2}$	$\frac{3}{2\cdot 2\cdot 2}$	$n=1$
$f''''(x) = \frac{-3\cdot 5}{2\cdot 2\cdot 2\cdot 2}x^{-7/2}$	$\frac{-3\cdot 5}{2\cdot 2\cdot 2\cdot 2}$	$n=2$

$\sqrt{x}$ series $= 1 + \frac{1}{2}(x-1) - \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5 \cdots (2n+1)}{2^{n+2}} (x-1)^{n+2}$

Radius $= ?$ Probably 1 

Accd to book's table, <u>is 1</u>

(It's Theorem 17 that it equals 1.
See exercise 75 for how to prove it.)

(I couldn't see how/if one might apply
Taylor's inequality.)